Examples of Irreducible Automorphisms of Handlebodies

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Abstract

Automorphisms of handlebodies arise naturally in the classification of automorphisms of three-manifolds. Among automorphisms of handlebodies, there are certain automorphisms called irreducible (or generic), which are analogues of pseudo-Anosov automorphisms of surfaces. We show that irreducible automorphisms of handlebodies exist and develop methods for constructing a range of examples.

1 Introduction.

1.1 Some history and background

The classification of *automorphisms* (i.e. self-homeomorphisms) of a manifold, up to isotopy, is a natural and important problem. Nielsen addressed the case where the manifold is a compact and connected surface and his results were later substantially improved by Thurston (see [Nie86a, Nie86b, Nie86c, Thu88, HT85]). We briefly state their main result: An automorphism of a surface is, up to isotopy, either *periodic* (i.e., has finite order), *reducible* (i.e., preserves an essential codimension-1 submanifold) or *pseudo-Anosov*. We refer the reader to any of [FLP79, HT85, Thu88, CB88] for details — including the definition of a pseudo-Anosov automorphism. The Nielsen-Thurston theory also shows that the reducible case may — as expected — be reduced to the other two. Since periodic automorphisms are relatively easy to understand, the remaining *irreducible* case — pseudo-Anosov — is the most interesting and rich one.

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Indeed, pseudo-Anosov automorphisms of surfaces are the subject of intense and wide research (see [Thu88]). We mention two works on the natural problem of building examples of such automorphisms: Penner provides a generating method [Pen88] and a testing algorithm is developed in [BH95].

In [Oer02], Oertel undertakes a similar classification project for a certain class of three-dimensional manifolds. Suppose that a three-dimensional manifold M is compact, connected, orientable and *irreducible* (i.e., every embedded sphere bounds a ball). Assume further that $\partial M \neq \emptyset$. By use of canonical decompositions of M due to Bonahon (determined by his *characteristic compression body* [Bon83]) and Jaco, Shalen and Johanson (the *JSJ-decomposition* [JS79, Joh79]), the study of automorphisms of M is reduced to the study of automorphisms of *compression bodies* and *handlebodies* (see [Oer02]). We define these types of manifolds:

Definition 1.1. A *handlebody H* is an orientable and connected three-manifold obtained from a three-dimensional ball by attaching a certain finite number g of 1-handles. The integer g is the *genus* of H. It should be clear that $\pi_1(H)$ is isomorphic to the free group F_g on g generators.

A *compression body* is a pair (Q, F), where Q is a three-manifold obtained from a compact surface F (not necessarily connected) in the following way: consider the disjoint union of $F \times I$ with the disjoint union of finitely many balls B and add 1-handles to $(F \times \{1\}) \cup \partial B$, obtaining Q. We allow empty or non-empty ∂F, but F cannot have sphere components. We identify F with $F \times \{0\} \subseteq Q$, which is called the *interior boundary* of (Q, F), denoted by $\partial_i Q$. The *exterior boundary* $\partial_e Q$ of Q is the closure $\overline{\partial Q - \partial_i Q}$. If Q is homeomorphic to the disjoint union of $F \times I$ with balls then (Q, F) is said to be *trivial*.

We may abuse notation and refer to Q as a compression body.

The role of the disjoint union of balls B in the definition of compression body above is a two fold one. It makes some operations of attaching 1-handles to be trivial. For instance, a 1-handle may connect $F \times I$ with a ball or connect two distinct balls. On the other hand, under this definition, a handlebody may be regarded as a connected compression body whose interior boundary $\partial_i Q = F$ is empty. Indeed, these two types of manifolds are quite similar [Bon83], as is the study of their automorphisms [Oer02]. This research will focus on the case of handlebodies.

The following definition is due to Oertel:

Definition 1.2. An automorphism $f \colon H \to H$ of a handlebody H is said *reducible* if any of the following holds:

1. there exists an f-invariant (up to isotopy) non-trivial compression body (Q, F) with $Q \subseteq H$, $\partial_e Q \subseteq \partial H$ and $F = \partial_i Q \neq \emptyset$ not containing ∂-parallel disc components,

2. there exists an f-invariant (up to isotopy) collection of pairwise disjoint, incompressible, non-∂-parallel and properly embedded annuli, or

3. H admits an f-invariant (up to isotopy) I-bundle structure.

The automorphism f is said *irreducible* (or *generic*, as in [Oer02]) if both of the following conditions hold:

1. $\partial f = f|_{\partial H}$ is pseudo-Anosov, and

2. there exists no *closed reducing surface F*: a closed reducing surface is a surface $F \neq \emptyset$ which is the interior boundary $\partial_i Q$ of a non-trivial compression body (Q, F) such that $Q \subseteq H$, (Q, F) is f-invariant (up to isotopy) and $\partial_e Q = \partial H$.

An obvious remark is that this definition of irreducible automorphism excludes the periodic case.

Theorem 1.3 (Oertel, [Oer02]). *An automorphism of a handlebody is either:*

1. *periodic,*

2. *reducible, or*

3. *irreducible.*

We note that the theorem above is not entirely obvious. For example, one must show that if an automorphism $f : H \to H$ of a handlebody does not restrict to a pseudo-Anosov ∂f on ∂H, then f is actually reducible according to Definition 1.2, or periodic.

Our interest is precisely in the irreducible case, which is in many ways analogous to the pseudo-Anosov case for surfaces (an important similarity is related to the existence of certain invariant projective measured laminations [Oer02]).

1.2   This research

In this article we address the problem of constructing examples of irreducible automorphisms of handlebodies. We note that no examples of such automorphisms were known before this research was undertaken. The examples will give some indication that the theory of irreducible automorphisms of handlebodies is even richer than the theory of pseudo-Anosov automorphisms of surfaces.

Our results will give sufficient conditions for an automorphism to be irreducible. These conditions will either be constructible or verifiable, so they can (and will) be used to generate actual examples.

In Section 2 we build an example of an irreducible automorphism of a genus two handlebody (see Example 2.1 and Proposition 2.2).

In Section 3 we generalize the construction of that first example and develop a method for generating a larger class of irreducible automorphisms. This is done in theorems 3.2 and 3.4. Their statements depend on some rather technical constructions, unsuited for this introduction.

The final section concerns closed reducing surfaces. Under some hypotheses, we will find bounds (upper and lower) for the Euler characteristic of a possible closed reducing surface. As a corollary we shall show:

(Corollary 4.4). *Let $f \colon H \to H$ restrict to ∂H as a pseudo-Anosov automorphism. If the genus of H is two then f is irreducible.*

One can use this as a tool to build examples of irreducible automorphisms. A consequence will be that the complexity of an irreducible automorphism of a handlebody cannot be extracted from the homomorphism induced in the fundamental group — unlike the analogous situation in dimension two. Example 4.5 will describe an irreducible automorphism of a handlebody H whose induced automorphism in $\pi_1(H) \to \pi_1(H)$ is the identity.

The following two theorems will serve us as important tools. We refer the reader to [Pen88] and [BH92] respectively for details and precise definitions.

Theorem 1.4 (Penner, [Pen88]). *Let $\mathcal{C}$, $\mathcal{D}$ be two systems of closed curves in an orientable surface S with $\chi(S) < 0$. Assume that $\mathcal{C}$ and $\mathcal{D}$ intersect efficiently, do not have parallel components and fill S. Let $f \colon S \to S$ be a composition of Dehn twists: right twists along curves of $\mathcal{C}$ and left twist along curves of $\mathcal{D}$. If a twist along each curve appears at least once in the composition, then f is isotopic to a pseudo-Anosov automorphism of S.*

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Theorem 1.5. *Let S be a compact surface with $\chi(S) < 0$ and precisely one boundary component. An automorphism $f \colon S \to S$ is pseudo-Anosov if and only if f_*^n is irreducible for all $n > 0$.*

As a final introductory remark, we observe that an ideal classification of automorphisms of handlebodies should identify in each isotopy class a representative which is "best" in some sense. Considering the classification of Theorem 1.3, this has been done for periodic and many reducible automorphisms [Bon83, Oer02]. The problem of finding a best representative of an irreducible automorphism is addressed in [Oer02, Car03] but not yet solved.

We will adopt the following notation: given a topological space A (typically a manifold or sub-manifold), $\overline{A}$ will denote its topological closure, $\mathring{A}$ its interior and $|A|$ its number of connected components. If M is a manifold and $S \subseteq M$ a compact codimension 1 submanifold, we can "cut M open along S" obtaining M_S. More precisely, a Riemannian metric in M determines a path-metric in $M - S$, in which the distance between two points is the infimum of the lengths of paths in $M - S$ connecting them. We let M_S to be the completion of $M - S$ with this metric.

I thank Ulrich Oertel for many enlightening meetings and helpful suggestions in his role as dissertation advisor, and also for laying the foundations on which the research in this paper is built.

2   An example.

We show that irreducible automorphisms of handlebodies exist by presenting an example.

Let H be a genus 2 handlebody. We will describe an automorphism of H as a composition of Dehn twists along two annuli and a disc. We shall prove that it is irreducible by showing that its restriction to ∂H is pseudo-Anosov and that, for an algebraic reason, there can be no closed reducing surface.

Example 2.1. We start with a pseudo-Anosov automorphism $\varphi \colon S \to S$ of the once punctured torus S. Such a φ will be defined as a composition of Dehn twists along two curves.

We will represent S as a cross, after identifying pairs of opposite sides as shown in Figure 1.

Let α_0, α_1 be simple closed curves as in the figure. It is easy to verify that the systems $\mathcal{C} = \{\alpha_0\}$ e $\mathcal{D} = \{\alpha_1\}$ satisfy the hypothesis of Theorem 1.4 (Penner). Let T_0^- be the left Dehn twist along α_0 and T_1^+ the right twist
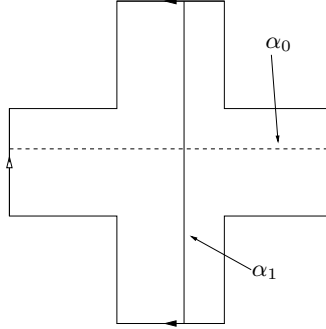
Figure 1: The oriented surface S and the curves α_0, α_1.

along α_1. We define:

$$\varphi = T_1^+ \circ T_0^-.$$

By Theorem 1.4, φ is pseudo-Anosov. Then, by Theorem 1.5, any positive power φ_*^n, of the induced homomorphism $\varphi_* \colon \pi_1(S) \to \pi_1(S)$ is irreducible. We note this fact for future use.

We now consider the handlebody $H = S \times I$ and lift φ to H, obtaining $\phi \colon H \to H$, a composition of twists along the annuli $A_0 = \alpha_0 \times I$, $A_1 = \alpha_1 \times I$ as in Figure 2.

Remark. For future use, we will think of the picture as being looked at "from above". More precisely, we orient H in such a way that the induced orientation in $S \times \{1\}$ coincides with the one inherited naturally from S.

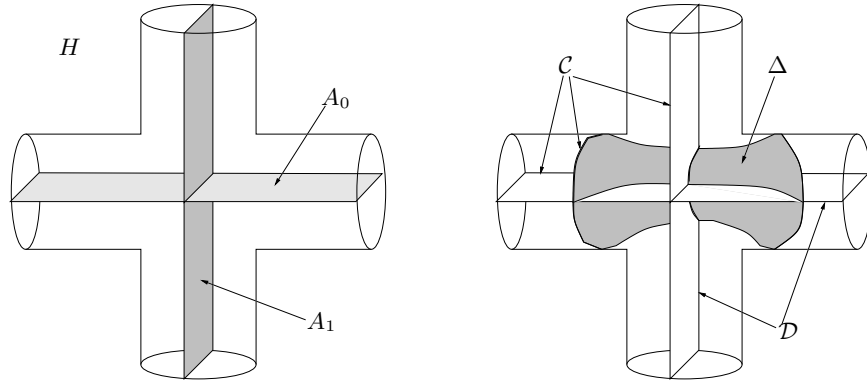Identifying $\pi_1(H)$ with $\pi_1(S)$ we have $\phi_* = \varphi_*$.

Figure 2: The automorphism f is defined as a composition of Dehn twists along the annuli A_0, A_1 and the disc Δ.

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Finally, we will obtain the desired irreducible automorphism $f \colon H \to H$ by composing ϕ with a twist along a disc Δ, shown in Figure 2.

Let T_Δ^+ be the right Dehn twist along Δ. We define:

$$f = T_\Delta^+ \circ \phi.$$

Proposition 2.2. *The automorphism $f \colon H \to H$ is irreducible.*

Part of the proof will be done in the following general lemma.

Lemma 2.3. *Let $g \colon H \to H$ be an automorphism of a handlebody H such that ∂g is pseudo-Anosov. If g is reducible then, for some $n \in \mathbb{N}$, $g_*^n \colon \pi_1(H) \to \pi_1(H)$ is reducible.*

Proof. Let Q be a compression body invariant under g. Let $F \subseteq \partial_i Q$ be a component of the closed reducing surface and $J \subseteq \mathring{H}$ the handlebody bounded by F. Choosing a base point in J and omitting the obvious inclusion homomorphisms we claim that

$$\pi_1(H) = \pi_1(J) * G,$$

where G is not trivial. To see this, first consider the connected and nontrivial compression body $Q' = \overline{H - J}$, whose boundary decomposes as $\partial_i Q' = F$ and $\partial_e Q' = \partial H$. The compression body structure of Q' gives it as a product $F \times I$ to which 1-handles are attached. Regarding $F \times I \subseteq Q' \subseteq H$, we see that the handlebody $J' = (F \times I) \cup J$ deformation retracts to J (so $\pi_1(J') = \pi_1(J)$ through inclusion). But the compression body structure of Q' gives H as J' with 1-handles attached to $\partial J'$. Since $\partial J'$ is connected, we can moreover assume that these 1-handles are attached to a disc in $\partial J'$, which gives $\pi_1(H) = \pi_1(J') * G = \pi_1(J) * G$, where G is a free group (whose rank equals the number of 1-handles of Q'). Since Q' is not trivial, G is not trivial, proving the claim. Therefore $\pi_1(J)$ is a proper free factor of $\pi_1(H)$.

Let g^n be the first power of g preserving J. Isotoping g we assume moreover that the base point is fixed by g^n. From

$$g^n(J) = J$$

follows that $g_*^n(\pi_1(J))$ is conjugate to $\pi_1(J)$, hence the class of g_*^n in $Out\big(\pi_1(H)\big)$ is reducible. $\qquad\square$

Proof of Proposition 2.2. We need to prove that $\partial f = f|_{\partial H}$ is pseudo-Anosov and that f does not admit closed reducing surfaces.

We start by verifying that ∂f is pseudo-Anosov. It is given as composition of Dehn twists: left twists along curves of

$$\mathcal{C} = \{\, (\alpha_0 \times \{1\}),\ (\alpha_1 \times \{0\}) \,\},$$

(see Figure 2) and right twists along curves of

$$\mathcal{D} = \{\, (\alpha_0 \times \{0\}),\ (\alpha_1 \times \{1\}),\ \partial\Delta \,\}.$$

We now note that $\mathcal{C}$, $\mathcal{D}$ satisfy the hypotheses of Theorem 1.4, hence ∂f is pseudo-Anosov.

We prove by contradiction that f admits no closed reducing surface. Suppose there is a closed reducing surface. By Lemma 2.3, there exists n such that f_*^n is reducible. But $f = (T_\Delta^+) \circ \phi$ and the twist (T_Δ^+) (along a disc) induces the identity in $\pi_1(H)$. Therefore, recalling that $\pi_1(H)$ is identified with $\pi_1(S)$, we have that $f_*^n = \phi_*^n = \varphi_*^n$, which was seen before to be irreducible for any n, a contradiction.

Therefore f is irreducible. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 A method for generating irreducible automorphisms.

The construction of Example 2.1 may be generalized to provide a method for generating a larger class of irreducible automorphisms of handlebodies (Theorems 3.2 and 3.4). This method partially solves a problem proposed in [Oer02].

Definition 3.1. We say that a pair $(\mathcal{C}, \mathcal{D})$ of curve systems in a compact, connected and orientable surface S with $\chi(S) < 0$ is a *Penner pair in S* if $\mathcal{C}$, $\mathcal{D}$ satisfy the hypotheses of Penner's Theorem 1.4 i.e.,

1. each $\mathcal{C}$, $\mathcal{D}$ is a finite collection of simple, closed and pairwise disjoint essential curves,

2. $\mathcal{C}$ and $\mathcal{D}$ intersect efficiently, do not have parallel components and *fill S* (i.e., the components of $S - (\mathcal{C} \cup \mathcal{D})$ are either contractible or deformation retract to ∂S).

Suppose that $(\mathcal{C}, \mathcal{D})$ is a Penner pair. An automorphism φ of S obtained from $\mathcal{C}$, $\mathcal{D}$ as in Theorem 1.4 is called a *Penner automorphism subordinate to $(\mathcal{C}, \mathcal{D})$*.

If $\partial S \neq \emptyset$ then a properly embedded and essential arc θ is called *dual to* $(\mathcal{C}, \mathcal{D})$ if θ intersects $\mathcal{C} \cup \mathcal{D}$ transversely and in exactly one point $p \notin \mathcal{C} \cap \mathcal{D}$.

Remark. Although not all Penner pairs admit dual arcs it is easy to construct pairs that do: such a pair $(\mathcal{C}, \mathcal{D})$ in S has the property that there are two adjacent components (not necessarily distinct) of $S - (\mathcal{C} \cup \mathcal{D})$ each containing some component of ∂S. If a pair does not have this property then we can remove discs from S and introduce dual arcs.

We constructed the irreducible automorphism in Example 2.1 by lifting a pseudo-Anosov automorphism of a surface to a product and composing it with a twist on a disc. The general method will be similar. Our interest in dual arcs is that we can use them to construct discs that will yield irreducible automorphisms.

Throughout this section we fix a compact, connected and oriented surface S with $\partial S \neq \emptyset$ and define $H = S \times I$, which is a handlebody. We identify S with $S \times \{1\} \subseteq H$, inducing orientation in H.

Given a Penner pair $(\mathcal{C}, \mathcal{D})$ in S and a dual arc θ, we build a disc Δ_θ in H in the following way. Let γ be the curve of $(\mathcal{C}, \mathcal{D})$ that θ intersects and assume without loss of generality that $\gamma \subseteq \mathcal{C}$. Let $D = \theta \times I \subseteq H$. Then ∂D intersects $\gamma_1 = \gamma \times \{1\}$ in a point. Now let Δ_θ be the *band sum* of D with itself along γ_1. This means that Δ_θ is obtained from D and γ_1 by the following construction: consider a regular neighborhood $N = N(D \cup \gamma_1)$. Then $\Delta_\theta = \overline{\partial N - \partial H}$ is a properly embedded disc.

Theorem 3.2. *Suppose that $\partial S \neq \emptyset$ has exactly one component. Let $(\mathcal{C}, \mathcal{D})$ be a Penner pair in S with dual arc θ and $\varphi \colon S \to S$ a Penner automorphism subordinate to $(\mathcal{C}, \mathcal{D})$. Let $\hat{\varphi} \colon H \to H$ be the lift of φ to the product $H = S \times I$ and $\Delta_\theta \subseteq H$ the disc constructed from the arc θ as above. Then there exists a Dehn twist $T_{\Delta_\theta} \colon H \to H$ along Δ_θ such that the composition*

$$\hat{\varphi} \circ T_{\Delta_\theta} \colon H \to H$$

is an irreducible automorphism of H.

The key to the proof is the verification that $\mathcal{C}$, $\mathcal{D}$ and $\partial \Delta_\theta$ induce a Penner pair in ∂H.

Lemma 3.3. *Let S, $(\mathcal{C}, \mathcal{D})$, θ, $H = S \times I$ and Δ_θ be as in the statement of Theorem 3.2. Let $\mathcal{C}_i = \mathcal{C} \times \{i\} \subseteq S_i = S \times \{i\}$ and $\mathcal{D}_i = \mathcal{D} \times \{i\} \subseteq S_i = S \times \{i\}$, defining $\mathcal{C}_0, \mathcal{D}_0 \subseteq S_0$ and $\mathcal{C}_1, \mathcal{D}_1 \subseteq S_1$. Under these conditions the following system of curves in ∂H:*

$$\begin{aligned}
\mathcal{Q} &= \mathcal{D}_0 \cup \mathcal{C}_1 \cup \{\partial \Delta_\theta\}, \\
\mathcal{R} &= \mathcal{C}_0 \cup \mathcal{D}_1,
\end{aligned}$$

determine a Penner pair $(\mathcal{Q}, \mathcal{R})$ in ∂H.

Proof. We start by making the obvious remarks that $\mathcal{C}_0$, $\mathcal{D}_0$, $\mathcal{C}_1$, $\mathcal{D}_1 \subseteq \partial H$ and $\mathcal{C}_0 \cap \mathcal{D}_1 = \emptyset$, $\mathcal{D}_0 \cap \mathcal{C}_1 = \emptyset$. Recall we are assuming that $\theta \cap (\mathcal{C} \cup \mathcal{D}) \subseteq \gamma \subseteq \mathcal{C}$. We verify that:

- $\partial \Delta_\theta \cap \mathcal{D}_0 = \emptyset$, because $(\theta \times \{0\}) \cap \mathcal{D}_0 = \emptyset$ and $\partial \Delta_\theta \cap S_0$ consists of two arcs parallel to $\theta \times \{0\}$,

- $\partial \Delta_\theta \cap \mathcal{C}_1 = \emptyset$, because $\partial \Delta_\theta \cap \gamma_1 = \emptyset$ by construction.

Therefore each $\mathcal{Q} = \mathcal{D}_0 \cup \mathcal{C}_1 \cup \{\partial \Delta\}$ and $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{D}_1$ is a system of simple closed curves essential in ∂H. To conclude that $(\mathcal{Q}, \mathcal{R})$ is indeed a Penner pair we just need to verify that $\mathcal{Q} \cup \mathcal{R}$ fills ∂H.

A component of $S - (\mathcal{C} \cup \mathcal{D})$ either is a disc or an annulus that retracts to ∂S. Therefore a component of $\partial H - (\mathcal{C}_0 \cup \mathcal{D}_0 \cup \mathcal{C}_1 \cup \mathcal{D}_1)$ either is a disc, or an annulus A (that retracts to $\partial S \times I$). But $A \cap \partial \Delta_\theta$ is a union of four arcs essential in A, hence each component of $\partial H - (\mathcal{Q} \cup \mathcal{R})$ is a disc, showing that $\mathcal{Q} \cup \mathcal{R}$ fills ∂H, completing the proof. $\square$

Instead of proving Theorem 3.2 we will prove the more general result below, which clearly implies the other. We note that twists on curves of $\mathcal{C}$, $\mathcal{D}$ in S lift to twists along annuli in H. We call these systems of annuli $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}$ respectively. It thus makes sense to refer to directions of the twists along these vertical annuli (recall that H has orientation induced by $S \times \{1\} \subseteq H$).

Theorem 3.4. *Let $(\mathcal{C}, \mathcal{D})$, S, θ, H and Δ_θ be as in Theorem 3.2. Let f be a composition $f \colon H \to H$ of twists along the annuli of $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}$ and the disc Δ_θ: in one direction along the annuli in $\hat{\mathcal{D}}$ and in the opposite direction along the annuli in $\hat{\mathcal{C}}$ and the disc Δ_θ. If each of these twists appear in the composition at least once f is irreducible.*

Proof. We will show initially that $f_*^n \colon \pi_1(H) \to \pi_1(H)$ is an irreducible automorphism of a free group for any $n \geq 0$ (hence there can be no closed reducing surface by Lemma 2.3) and then that $\partial f = f|_{\partial H}$ is pseudo-Anosov, thus completing the proof that f is irreducible.

We first identify $\pi_1(H)$ with $\pi_1(S)$, identifying S with $S \times \{1\} \subseteq H$. Let T_{Δ_θ} be a twist along Δ_θ. Since $(T_{\Delta_\theta})_* \colon \pi_1(H) \to \pi_1(H)$ is the identity (Δ_θ is a disc) the hypotheses on f imply that $f_* = \varphi_*$ for some Penner automorphism $\varphi \colon S \to S$ subordinate to $(\mathcal{C}, \mathcal{D})$. Penner automorphisms are pseudo-Anosov so, given that ∂S has a single component, it follows from Theorem 1.5 that φ_*^n is an irreducible automorphism of $\pi_1(S)$ for any $n \geq 0$. Therefore $f_*^n \colon \pi_1(H) \to \pi_1(H)$ is irreducible, proving that f does not admit closed reducing surfaces (Lemma 2.3).

We now prove that ∂f is pseudo-Anosov. Let $(\mathcal{Q}, \mathcal{R})$ be as in Lemma 3.3, therefore a Penner pair. By construction the twists that compose f restrict to ∂H as twists along curves of $\mathcal{Q}$ or $\mathcal{R}$. It is then straightforward to verify that ∂f is a Penner automorphism subordinate to $(\mathcal{Q}, \mathcal{R})$, hence pseudo-Anosov, completing the proof that f is irreducible. $\square$

Example 3.5. Consider S a genus 2 surface minus a disc, represented in Figure 3 as an octagon whose sides are identified according to the arrows.
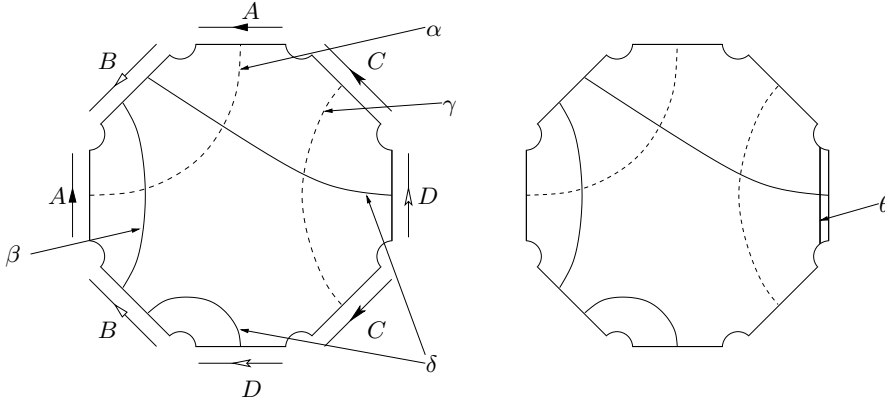
Figure 3: A Penner pair in S, with dual arc θ.

In the picture there are represented four further curves: α, β, γ and δ. Defining

$$\mathcal{C} = \{\, \beta, \delta \,\},$$
$$\mathcal{D} = \{\, \alpha, \gamma \,\},$$

it is easy to check that $(\mathcal{C}, \mathcal{D})$ is a Penner pair in S. The automorphism $\varphi \colon S \to S$ defined by

$$\varphi = T_\beta^- \circ T_\delta^- \circ T_\alpha^+ \circ T_\gamma^+$$

is, therefore, a Penner automorphism subordinate to the pair $(\mathcal{C}, \mathcal{D})$.

The pair $(\mathcal{C}, \mathcal{D})$ admits dual arcs. The picture shows one, labelled as θ. We consider the corresponding disc Δ_θ. Figure 4 shows $S_0 = S \times \{0\}$, $S_1 = S \times \{1\} \subseteq \partial H$ and how $\partial \Delta_\theta$ intersects them.

Figure 4 shows the pair $(\mathcal{Q}, \mathcal{R})$ obtained by Lemma 3.3 as well: $\mathcal{Q}$ consists on the solid curves, including $\partial \Delta_\theta$, while the dotted curves form $\mathcal{R}$.

Theorem 3.2 assures that, if $\hat{\varphi} \colon H \to H$ is the lift of φ to H, then

$$\hat{\varphi} \circ T_{\Delta_\theta} \colon H \to H$$

is an irreducible automorphism for a certain twist T_{Δ_θ} along Δ_θ.
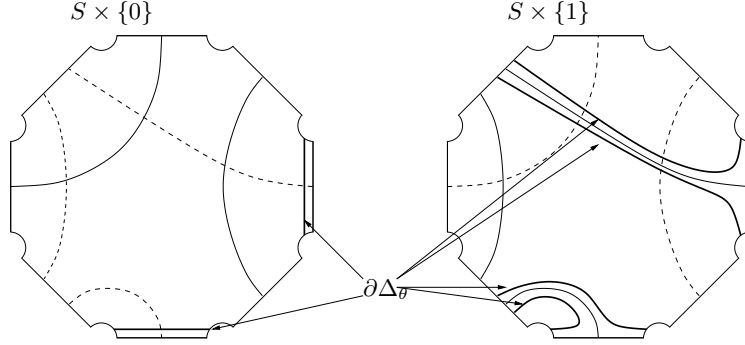
Figure 4: The curve $\partial\Delta_\theta$ in ∂H.

4 Genus of a closed reducing surface.

Consider an irreducible automorphism f of a handlebody H. By definition, the restriction $\partial f = f|_{\partial H}$ is pseudo-Anosov. We prove in this section that, for genus 2 handlebodies, the converse is also true: if ∂f is pseudo-Anosov f is irreducible (Corollary 4.4). Therefore methods for generating pseudo-Anosov automorphisms may be used for generating irreducible automorphisms of handlebodies (e.g., Example 4.5). The nonexistence of a closed reducing surface in this case comes from a geometric argument: see Theorem 4.3, which determines bounds on the Euler characteristic of a closed reducing surface. Therefore this criterion does not depend on the automorphism induced on the fundamental group and, for this reason, yields interesting examples, even if the hypothesis on the genus looks restrictive.

The following lemma is elementary, so we leave the proof to the reader.

Lemma 4.1. *Let (Q, F) be a connected and non-trivial compression body with $F \neq \emptyset$. If n is the smallest number of 1-handles of Q then*

$$\chi(\partial_e Q) = \chi(F) - 2n.$$

Proposition 4.2. *Let (Q, F) be a non-trivial compression body with only one 1-handle and $F \neq \emptyset$. Then the handle is unique up to isotopy.*

Proof. We start by noting that, up to isotopy, the 1-handle and a *dual disc* (the co-core of a 1-handle) determine each other.

We fix a 1-handle for (Q, F) and a dual disc D. Consider another choice of 1-handle and pick a dual disc D'. We can assume without loss of generality that Q is connected: Q has just one 1-handle so all but one of the components

(the one containing the initial 1-handle) are either products or balls. Since $\partial_e Q \cap L$ is incompressible in such a product (or ball) component L, the component containing the initial 1-handle must contain both D and D', hence the other 1-handle as well.

We shall show that D and D' are isotopic. As mentioned previously, this implies the proposition. The proof will be done in two steps. The first step simplifies $D \cap D'$ through standard "cutting and pasting" methods, obtaining disjoint D and D'. In the second step we will show that D and D' must be parallel, as desired.

We begin the first step by perturbing D' so that $D \cap D'$ consists of closed curves and arcs. We can eliminate closed curves from $D \cap D'$ by standard "cut and paste" arguments. Since Q is irreducible this can be attained by isotopy of D'. After a finite number of such operations we have that $D \cap D'$ contains no closed curves.

So assume that $D \cap D'$ consists of arcs. Here again we will perform isotopies that will reduce $|D \cap D'|$. Recall that $F = \partial_i Q$ and consider two cases: 1, that F is disconnected, and 2, that F is connected.

1. F is disconnected. In this case D separates Q. Let α be an arc of $D \cap D'$ which is edgemost in D', cutting a half-disc Δ' from D'.

Let $Q' = \overline{Q - D}$ with product structure inherited from Q. Since D separates Q it follows that Q' has two components (see Figure 5).
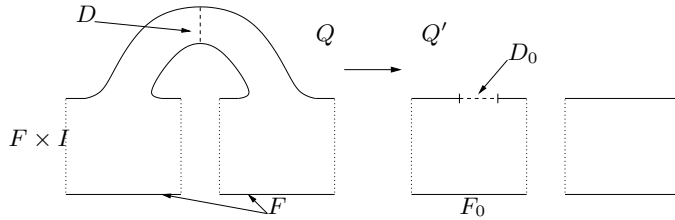

Figure 5: Cutting Q open along D yields Q'.

Let $F_0 \times I$ be the component of Q' containing Δ'. There exists a disc $D_0 \subseteq F_0 \times \{1\}$ corresponding to D and, abusing notation, let $\alpha \subseteq D_0$ be defined by $\alpha = \partial \Delta' \cap D_0$. Moreover, we have that $\partial \Delta' \subseteq F_0 \times \{1\}$. Since $F_0 \times \{1\}$ is incompressible, $\partial \Delta'$ must bound a disc $\Delta'' \subseteq F_0 \times \{1\}$. By the choice of Δ', $\Delta'' \cap D_0$ is a disc Δ'''. Irreducibility of Q' implies that the sphere $\Delta' \cup \Delta''$ bounds a ball. Regluing Q' to recover Q, such a ball determines another ball B in Q. We isotope Δ' through B a little beyond Δ''' to remove α from $D \cap D'$, thus reducing $|D \cap D'|$.

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A finite sequence of such isotopies yields $D \cap D' = \emptyset$.

2. F is connected. Now D does not separate Q. Consider again an arc $\alpha \subseteq D \cap D'$ which is edge-most in D', bounding with an arc $\beta \subseteq \partial D'$ an edge-most disc $\Delta' \subseteq D'$.

Let $Q' = \overline{Q - D} \simeq F \times I$ be the compression body Q cut open along D (Figure 6).
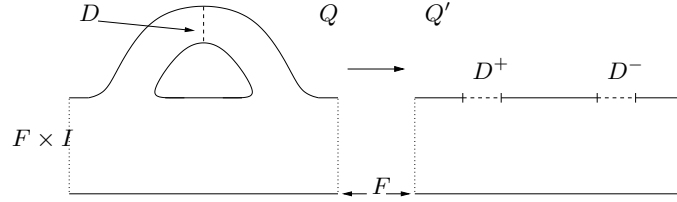

Figure 6: Cutting Q open along D yields Q'.

In this case $F \times \{1\}$ contains two discs D^+, D^- corresponding to D and we suppose that D^+ corresponds to the side of D associated to Δ'. There are two arcs, $\alpha^+ \subseteq D^+$ and $\alpha^- \subseteq D^-$ corresponding to α. Proceeding as in the previous case, we have that $\partial \Delta' \cap D^+ = \alpha^+$, $\partial \Delta' \cap D^- = \emptyset$, $\partial \Delta' \subseteq F \times \{1\}$. Again incompressibility gives us $\partial \Delta'$ bounding a disc $\Delta'' \subseteq F \times \{1\}$. There are two possibilities to be considered depending on $\Delta'' \cap D^-$:

(a) If $\Delta'' \cap D^- = \emptyset$ (see Figure 7 (a), recalling that $\alpha \cup \beta = \partial \Delta' = \partial \Delta''$) we proceed as in case 1, removing the arc α from $D \cap D'$, reducing $|D \cap D'|$.

(b) If $\Delta'' \cap D^- \neq \emptyset$ (Figure 7 (b)) the argument is more laborious. Since $\partial \Delta' \cap D' = \emptyset$, in this case we must have $\Delta'' \cap D^- = D^-$.
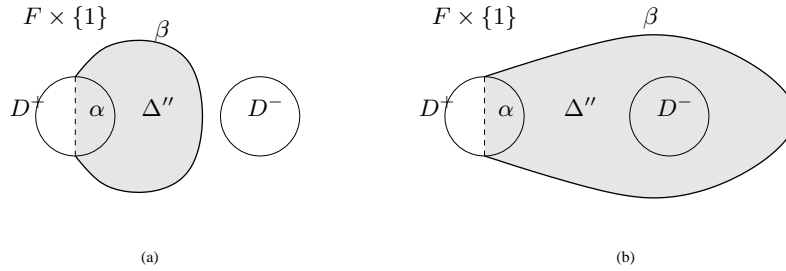

Figure 7: The disc $\Delta'' \subseteq F \times \{1\}$ parallel to Δ'. The cases (a) and (b) depend on $\Delta'' \cap D^-$.

Back to considering D, $D' \subseteq Q$, we note that $\partial D' \cap D$ is finite. Therefore, the set $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ of the closures of the components of $\partial D' - D$ consists of finitely many arcs. We fix an arbitrary orientation for $\partial D'$ and

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use it to induce a cyclic order in Γ. We can assume that the indices respect this order and, since $\beta \in \Gamma$, we can assume further that $\gamma_0 = \beta$ (see Figure 8).
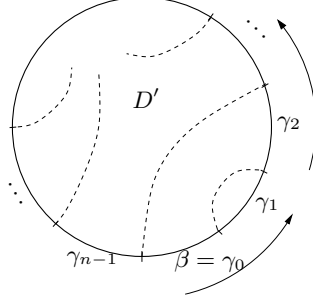

Figure 8: Orientation of the arcs in $\partial D'$.

In addition to the order in Γ, the orientation of $\partial D'$ induces an orientation in each γ_i, giving an order between the ends of each arc.

Working in Q' again, assume that some arc γ_i of Γ has both extremes in ∂D^-. In this case γ_i, together with an arc in ∂D^-, bounds a disc $\Delta''' \subseteq \overline{(F \times \{1\} - (D^+ \cup D^-))}$ for, in Q', $\gamma_i \cap D^+ = \emptyset$, $\gamma_i \cap \beta = \emptyset$ (see Figure 9).
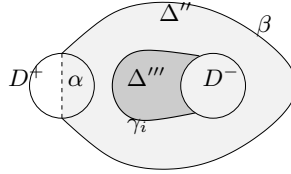

Figure 9: The disc Δ''' used to simplify $D \cap D'$.

Once again in Q, an isotopy may be use to pull γ_i and the whole $\partial D' \cap \Delta'''$ along Δ''' through D. This process either

- maintains $|\Delta \cap D|$ unchanged removing an arc of $\Delta \cap D$ but introducing a closed curve, which we remove using the argument previously presented, or

- two arcs of $\Delta \cap D$ are joined to one.

In both situations we reduce the number of components (all arcs) of $D \cap D'$.

It remains to consider the case when no γ_i has both points in ∂D^-. We shall show that this case cannot happen.

Recall that the ends of each γ_i have an order induced by the orientation in $\partial D'$ and note that $\gamma_0 = \beta$ has both ends in ∂D^+. It follows that γ_1 has its first end in ∂D^-. Since no γ_i has both ends in ∂D^- the final end of γ_1 is in ∂D^+. The same argument shows that if γ_i goes from ∂D^- to ∂D^+ then so does γ_{i+1}, therefore, by induction, all γ_i, $i \geq 1$ have this property. But Γ is finite with cyclic order, hence some $\gamma_n = \gamma_0 = \beta$, which does not have this property, a contradiction.

This completes the analysis of case 2, showing that, also in this situation, we can choose D' in its isotopy class in such a way that $D \cap D' = \emptyset$.

The above case analysis shows that we can always assume that D' is disjoint from D. It remains to prove that D and D' are parallel, hence isotopic. Again we divide the argument into the same two cases:

1. F is disconnected (D separates Q). Let Q_0 be the component of Q' (Q cut open along D) containing D'. As before, $Q_0 \simeq F_0 \times I$ and there is a disc $D_0 \subseteq F_0 \times \{1\}$ corresponding to D. Since $\partial D' \subseteq F_0 \times \{1\}$, incompressibility of $F_0 \times \{1\}$ and irreducibility of $F_0 \times I$ imply that D' is parallel to a disc $\Delta \subseteq F_0 \times \{1\}$. There are only two possibilities: either $\Delta \cap D_0 = \emptyset$ or $\Delta \cap D_0 = D_0$. If $\Delta \cap D_0 = \emptyset$ then D' is parallel to $\partial_e Q$ in Q, hence it is not a compressing disc for $\partial_e Q$, a contradiction. Therefore $\Delta \cap D_0 = D_0$, and $\Delta - D_0$ is an annulus, showing that D' parallel to D_0. Going back to Q, D' is parallel to D, therefore they are isotopic.

2. F is connected (D does not separate Q). The argument is analogous to the one in the first case: we consider $Q' \simeq F \times I$, with two discs $D^+, D^- \subseteq Q'$ corresponding to D. Hence $D' \subseteq Q'$ is parallel to $\Delta \subseteq F \times \{1\}$ and the case $\Delta \cap (D^+ \cup D^-) = \emptyset$ cannot happen either. The cases $\Delta \cap (D^+ \cup D^-) = D^+$ or $\Delta \cap (D^+ \cup D^-) = D^-$ again give D' parallel to D in Q (hence isotopic). The only situation not analogous to the previous case is when $\Delta \cap (D^+ \cup D^-) = D^+ \cup D^-$. Here, as can be seen by going back to Q, D' separates Q, implying that F is not connected, a contradiction. $\square$

Theorem 4.3. *Let $f \colon H \to H$ be a reducible automorphism of a handlebody H. If $\partial f = f|_{\partial H}$ is pseudo-Anosov, then a closed reducing surface F satisfies*

$$\chi(\partial H) + 4 \leq \chi(F) \leq 0.$$

Proof. Indeed, since ∂f is pseudo-Anosov, f either is irreducible or, as is the case, admits a closed reducing surface. A closed reducing surface $F \subseteq \mathring{H}$ is the interior boundary of a non-trivial compression body (Q, F) which is f-invariant and whose exterior boundary is $\partial_e Q = \partial H$. Since ∂H is connected then so is $\partial_e Q$ and Q also. By Lemma 4.1

$$\chi(F) = \chi(\partial H) + 2n,$$

where $n \geq 1$ is the smallest number of 1-handles of Q. Let D be the disc dual to a 1-handle. If $n = 1$ then Proposition 4.2 above implies that $f(D) = D$, hence $f(\partial D) = \partial D \subseteq \partial H$, contradicting the hypothesis that ∂f is pseudo-Anosov. Therefore $n \geq 2$. It is clear that $\chi(F) \leq 0$ for, by definition, spheres are not reducing surfaces. $\square$

Corollary 4.4. *Let $f \colon H \to H$ restrict to ∂H as a pseudo-Anosov automorphism. If the genus of H is two then f is irreducible.*

Proof. If there were a closed reducing surface F for f then $2 \leq \chi(F) \leq 0$. $\square$

Remark. This result enables us to reduce the problems of identification or construction of irreducible automorphisms of genus two handlebodies to the better understood analogues for pseudo-Anosov automorphisms of surfaces. For instance, Penner's Theorem 1.4 [Pen88] becomes a method for generating irreducible automorphisms (e.g. Examples 2.1 and 4.5 below). Moreover Bestvina and Handel's algorithm to decide whether a given surface automorphism is pseudo-Anosov or not [BH95] may be used as an algorithm to decide whether an automorphism of a genus two handlebody is irreducible or not.

On the other hand the same result exposes differences between the two dimensions (see the remark after Example 4.5).

Example 4.5. Consider H a genus 2 handlebody. Figure 10 shows two curves C_0, C_1 in ∂H.

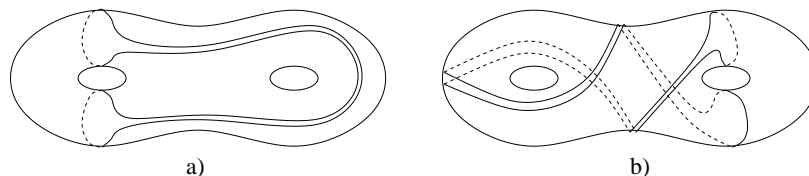a)   b)

Figure 10: a) the curve C_0 bounding the disc $D_0 \subseteq H$; b) the curve C_1 bounding the disc $D_1 \subseteq H$.

It is easy to see that these curves bound discs in H, D_0 and D_1 respectively (one can see them in the picture as the band sum of discs dual to the handles). We will define $f \colon H \to H$ as the composition of Dehn twists along these discs, to the left along D_1 and to the right along D_0:

$$f = T_{D_0}^+ \circ T_{D_1}^-.$$

It is routine to verify that $(\{C_0\}, \{C_1\})$ is a Penner pair in ∂H, hence

$$\partial f = T_{C_0}^+ \circ T_{C_1}^-$$

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is pseudo-Anosov (Theorem 1.4). By Corollary 4.4, f is irreducible.

Remark. The example of irreducible automorphism $f\colon H \to H$ above is given as a composition of twists on discs. Since twists on discs induce the identity on the fundamental group, it is immediate that $f_*\colon \pi_1(H) \to \pi_1(H)$ is the identity. Therefore, for a general irreducible automorphism f, f_* may fail to capture its complexity. We note that that is not the case for pseudo-Anosov automorphisms of surfaces. This difference should not be regarded as weakening the analogy between these two classes of automorphisms, but rather as exposing the richness of the three-dimensional setting.

The example below shows that neither inequalities of Theorem 4.3 can be improved.

Example 4.6. As in Example 4.5, Figure 11 a), b) represent the boundaries of two discs in a handlebody H, here with genus 3. These boundaries yield a Penner pair in ∂H as well. Hence a composition f of twists to opposite directions along these discs yields ∂f as a pseudo-Anosov automorphism. Theorem 4.3 says that the Euler characteristic of a closed reducing surface is zero. Indeed, one can see that there exists a torus that does not intersect the discs – therefore being invariant under f (Figure 11 c)).
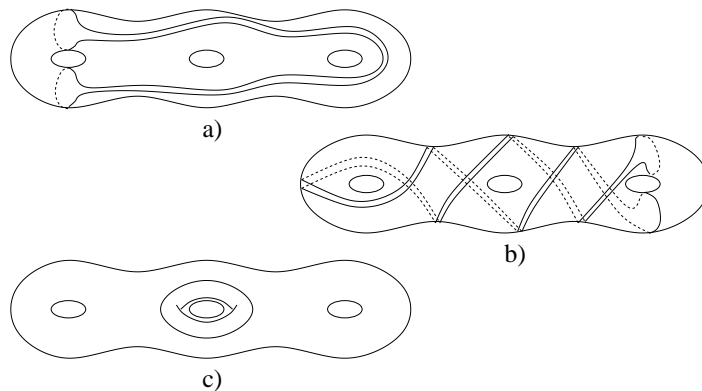

Figure 11: a) the curve C_0 bounding a disc $D_0 \subseteq H$; b) the curve C_1 bounding a disc $D_1 \subseteq H$; c) the invariant torus.

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